Solving quadratic equations – 3

Fundamental Theorem of Algebra and Vieta's formula

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This article is suitable for students in seventh grade and above and their parents, primary school teachers, middle school teachers, young adults, and any mathematics educator who wants to learn elementary mathematics again.

In "Solving Quadratic Equations - 2" , we mentioned: If we are told that there is a number in the form a+bi (the general form of complex numbers, where, a, b are two real numbers), whose square is i, then we can calculate what number a is and what number b is.

We just need to use the algebra operation rules and the property that the square of i is -1. By definition, we can calculate

$$i = (a + bi)^2 = (a + bi)(a + bi) = a^2 - b^2 + 2abi\,.$$

Comparing the real and imaginary parts of the two complex numbers on the left and right (the definition of two complex numbers being equal) we get

$$a^2 - b^2 = 0,\quad 2ab = 1\,.$$

From the above, we obtain: $a = b = \frac{\sqrt{2}}{2}$, or $a = b = -\frac{\sqrt{2}}{2}$. Thus there are two square roots of I :

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \quad and \quad -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\,.$$

Using the same idea, we can verify: for any complex number a+bi, we can find two complex numbers z, such that the squares of the two complex numbers are the complex number a+bi! In other words: there are always two pairs of real numbers x, y, they satisfy the equation

$$(x + yi)^2 = a + bi\,.$$

Combined with the method of completing the square we mentioned in "Solving to Quadratic Equations - 2", we can now say: Any quadratic equation with complex numbers as coefficients has two complex roots! In other words, any quadratic polynomial of one variable with complex numbers as coefficients can be factorized into the multiplication of two linear polynomials in the complex field - this is exactly the fundamental theorem of algebra for the quadratic equation of one variable!

Factorize a quadratic polynomial of one variable in the complex field into the multiplication of two linear polynomials

$$x^2 + px + q = (x - x_1)(x - x_2),$$

Where x_1, x_2 are two solutions to $x^2 + px + q = 0$. Multiplying the right-hand side of the above equation and comparing it with the corresponding term on the left-hand side, we will find the following rule (the famous Vieta's formula):

$$x_1 + x_2 = -p, \qquad and \quad x_1 \cdot x_2 = q\,.$$

Vieta's formula has wonderful scientific significance: for a given quadratic equation, we may not be able to write its two solutions (technically too difficult: for example, some students have not learned complex numbers or even real numbers), but we can get partial information on the solutions: we know what is the sum of two solutions, and we also know what is the product of two solutions!

You may say: After learning complex numbers, we can just solve the quadratic equation. Why bother using Vieta's formula to find the sum of two solutions and the product of two solutions? Yes, quadratic equations are nothing to be afraid of. But what about cubic equations, quartic equations, and even higher order equations? The derivation of Vieta's formula for quadratic equations has very good guiding significance for us.

Let us first consider the following question:

Can a n-degree polynomial be factorized into the product of n first degree polynomials? In other words: are there always n solutions to a n-th degree polynomial equation

$$x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0 = 0\ ?$$

The answer is yes (we have already proved it when n=2!) ---A n-th degree polynomial equation of one variable always has n roots (there may be duplicate roots). This result is called the Fundamental Theorem of Algebra for polynomial equations of one variable! We need to remind everyone that the proof of the Fundamental Theorem of Algebra will not be given until a college math course, called complex function. However, using the fundamental Theorem of Algebra, we can then derive the Vieta's formula just like the quadratic equation.

First note that the following equation always holds:

$$x^n + C_{n-1}x^{n-1} + \cdots + C_1x + C_0 = (x - x_1)(x - x_2) \cdots\cdots (x - x_n),$$

Where $x_1, x_2, \ldots, x_n$ the n roots to $x^n + C_{n-1}x^{n-1} + \cdots + C_1x + C_0 = 0$. From the above, through algebraic operations we obtain the Vieta's formula on the relationship between roots and coefficients. Here we list two of the relationships:

$$x_1 + x_2 + \cdots + x_n = -C_{n-1}$$

and

$$x_1 \cdot x_2 \cdots x_n = (-1)^n C_0$$

It is generally not easy to solve high-order equations for all complex roots. Thus partial t information given by Vieta's formula is very meaningful. If you don't believe it, you can try the following exercise.

Exercise. If x_1, x_2, x_3 are three solutions to $x^3 + 7x^2 - 9x + 13 = 0.$, find the value of

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}.$$